Benefits provided by partitions of unity with high regularity in crack modeling through enrichment procedures

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Motivation

GFEM-$C^0$

GFEM-$C^k$
Presentation topics

- Continuous partition of unity with $C^k$-GFEM
- Defining an approximation subspace
- Enrichment patterns and convergence rates
- Quality assessment through global measures
- Configurational forces method
- Quality assessment through local measures
- Smoothness, enrichments and conditioning
- Some improvements beyond...
C\(^\infty\) partition of unity – convex clouds

- No shape restrictions
- No coordinate mapping
- Flat-top property
- Simple numerical integration
- Blending

Edwards, *C\(^\infty\) finite element basis functions*, Report 45, Institute for Computational Engineering and Sciences – The University of Texas at Austin, 2006
\[ \xi_j (x) = n_{\alpha,j} \cdot (x - b_{\alpha,j}) \]

\[ k = \infty \quad \text{No restriction of patch shape} \]
\[ k = p - 1 \quad \text{Free of coordinate mapping!} \]

Edwards, \textit{C\textsuperscript{\infty} finite element basis functions}, Report 45, Institute for Computational Engineering and Sciences – The University of Texas at Austin, 1996


Barcellos, Mendonça and Duarte, \textit{A C\textsuperscript{k} continuous generalized finite element formulation applied to laminated Kirchhoff plate model}. Computational Mechanics, 44 (2009)
Continuous partition of unity with GFEM

Defining an approximation subspace

Quality assessment through global measures

Eshelbian mechanics

Quality assessment through local measure

Cloud-based residual error estimation

Some improvements beyond

\[ \mathcal{W}_\alpha (x) := \prod_{j=1}^{M_\alpha} \varepsilon_{\alpha,j} (\xi_j) \]

\[ \varphi_\alpha (x) = \frac{\mathcal{W}_\alpha (x)}{\sum_{\beta(x)} \mathcal{W}_\beta (x)}; \]

\[ \beta (x) \in \{ \gamma \mid \mathcal{W}_\gamma (x) \neq 0\} \]
Galerkin approximation

\[ u_p(x) = \sum_{\alpha=1}^{N} \varphi_\alpha(x) \left\{ u_\alpha + \sum_{i=1}^{q_\alpha} L_{\alpha i}(x) b_{\alpha i} + \sum_{j=1}^{q^s_\alpha} L^s_{\alpha j} b^s_{\alpha j} \right\} \]

if \( p=3 \)

\[ L_{\alpha 9}(x, y) = \left\{ \bar{x}, \bar{y}, \bar{x}^2, \bar{x} \bar{y}, \bar{y}^2, \bar{x}^3, \bar{x}^2 \bar{y}, \bar{x} \bar{y}^2, \bar{y}^3 \right\} \]

e.g.

\[ \bar{x} := \frac{x - x_\alpha}{h_\alpha} \]

Quality assessment through global measures

Configurational forces method

Quality assessment through local measures

Smoothness, enrichments and conditioning

Some improvements beyond...
Defining the degree of an approximation

\[ b = p + 1 \quad \text{for } C^0 \text{ PoU (conventional tent FEM shape function)} \]

\[ b = p \quad \text{for } C^k \text{ PoU} \]

\[ b = \text{degree of reproducible polynomial} \]

\[ p = \text{degree of polynomial enrichment} \]


Enrichment pattern $X$ convergence rates

Topologic enrichment


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Convergence in terms of global values

(a) Rate $C_k > rate \ C_0$;
(b) Error $C_k < error \ C_0$ for $b=1,2,3,4$.
Local measure using configurational forces


Variational balance of material linear momentum

\[ \Sigma (u) = \mathcal{W} (u) \mathbb{I} - \mathbb{L}^T (u) \sigma (u) \]

Eshelby tensor

\[ \Sigma = \{\Sigma_x, \Sigma_y, \Sigma_{xy}, \Sigma_{yx}\}^T \]

\[ \mathcal{W} = \frac{1}{2} \sigma^T \varepsilon \]

\[ \int_{\Omega} (\mathbb{L} \mathbf{v})^T \Sigma l_z \, d\Omega = \int_{\Omega} (\mathbf{v})^T \varrho l_z \, d\Omega \]

Inhomogeneity force

\[ \varrho = \{\varrho_x, \varrho_y\}^T \]

\[ \mathbf{v} = \{v_x, v_y\}^T \]

\[ \mathbb{L}(u) = \mathbb{L} \mathbb{I} u \quad \mathbb{I}^T = \{1, 1, 0, 0\} \]

\[ \mathbb{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \end{bmatrix} \quad \mathbb{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \quad \mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]
Convergence in J-integral

Geometric enrichment

(a) Rate $C_k > rate C^0$;
(b) Error $C_k < error C^0$ for $b=1,2,3,4$. 

GFEM-C$^k$
Topologic enrichment pattern

M1

M2

M3

M4

Branch functions and $p$-enrichment on green nodes

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Convergence in terms of global and local values

Topologic enrichment

Energy

J-integral

Quality assessment through global measures

Configurational forces method

Quality assessment through local measures

Smoothness, enrichments and conditioning

Some improvements beyond...

C0

Ck

M1, M2, M3, M4

p – convergence!
Convergence rates for $h$-refinement

<table>
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<tr>
<th>enrichment pattern</th>
<th>PoU</th>
<th>$b = 1$</th>
<th>$b = 2$</th>
<th>$b = 3$</th>
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<td>$C^0(\Omega)$</td>
<td>0.33</td>
<td>0.91</td>
<td>1.42</td>
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</tbody>
</table>
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**Condition number Nc**

**Geometric enrichment**

\[ \text{cond. number} = \frac{\text{larger eigenvalue}}{\text{smaller eigenvalue} \neq 0} \]

\[ Nc \ C^k > Nc \ C^0 \]

**GFEM-C^k**

**GFEM-C^0**

M1, M2, M3, M4
Eigenvalues distribution X enrichment

- Geometric pattern;
- Uniform p-enrichment

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Eigenvalues distribution \( X \) enrichment

**Topologic enrichment**

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Exact error dispersion \((u_y - u_{yh})\)

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Concluding remarks

- Continuous stress fields around singularity provide better severity crack parameters
- Polynomial enrichments together with branch functions may adaptively improve the stress fields
- Continuity may conduct to better computation of nodal Eshelby forces
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